GAME THEORY APPROACH TO SLIDING WINDOW H_∞ ALGORITHM FOR DYNAMICAL MODELING DESIGN

Jakub Dokoupil, Vladimír Burlak

Doctoral Degree Programme (2, 1), FEEC BUT

E-mail: xdokou05@stud.feec.vutbr.cz, xburla00@stud.feec.vutbr.cz

Supervised by: Petr Pivoňka

E-mail: pivonka@feec.vutbr.cz

Abstract: In this paper, we extend the game theory in order to adaptive process identification through a linear hypermodel. Deterministic cost function has been performed by using game theory approach, which enables us, with dynamic optimization technique, the derivation of recursive H_∞ method (RecH_∞) and sliding window recursive H_∞ method over finite data horizon (SW-RecH_∞). Advantages of the proposed approach are illustrated in the simulations example on the mathematical model.

Keywords: Game theory, recursive identification, H_∞ parameter estimation, sliding data horizon

1. INTRODUCTION

Majority of dynamical systems appearing in technical applications are time-varying, non-linear processes. The problem of adaptation to time varying parameters is arising in the case of their approximation with linear models, eventually for changing the power level of process. The ability of tracking time-varying parameters in recursive linear least-squares methods can be interpreted either in the stochastic context as an uncertainty increase in estimated parameters, eg. [5], or in the deterministic context as suppressing the influence of old data to new parameter estimates. One of the possibilities how the desired properties can be achieved, is to train the model on a sliding window over a finite data horizon [3,4].

The integration of finite data horizon technique into parameters optimization according to H_∞ , is enabled by game theory approach. Game theory has already been used for the H_∞ state estimation, see [1,2]. However, the contribution of this paper, which is the derivation of SW-RecH_∞ method, has not been yet, reported in the literature.

2. SLIDING WINDOW H_∞ ALGORITHM

Suppose hypermodel representing the behavior of a dynamic SISO process in the form

$$\theta_{k+1} = \theta_k \tag{1}$$

$$y_k = \varphi^T_{(k,\theta_k)}\theta_k + \varepsilon_{(k,\theta_k)} \tag{2}$$

where $\theta_k \in \mathbb{R}^n$ is the parameter vector, $\varphi^T_{(k,\theta_k)} \in \mathbb{R}^n$ is the regression vector, depending on θ_k in the case of pseudo linear regression, $\varepsilon_{(k,\theta_k)}$ is the prediction error value, ideally, having the properties of a zero mean white noise with the known covariance matrix R_k, ie. $\varepsilon_{(k,\theta_k)} \sim (0, R_k)$ and y_k is the measured output value from the process. For the optimal ARX model predictor, individual elements of vectors are filled according to

$$\varphi_k = \begin{bmatrix} u_{k-1} & \cdots & u_{k-n_b} & -y_{k-1} & \cdots & -y_{k-n_a} \end{bmatrix}^T \tag{3}$$

$$\theta_k = [b_{1k} \quad \cdots \quad b_{nbk} \quad a_{1k} \quad \cdots \quad a_{nak}]^T \tag{4}$$

where $u(k)$, $y(k)$ are observed input-output pairs of the process with orders of n_b, n_a.

The requirement is to find the value of the parameter vector $\hat{\theta}_k$ throughout the time span up to and including time $(N-1)$, but based only on p measurements in the time interval from $(N-p)$ to $(N-1)$, so on the basis of sliding rectangular data window. From a theoretical perspective, game strategy of optimization parameter model is understood as a competition of $\hat{\theta}_k$ against $\varepsilon_{(k,\theta_k)}$ and initial value θ_0 in order to minimize (against to maximize) the cost function V_1

$$V_1(\hat{\theta}_k, \varepsilon_{(k,\theta_k)}, \theta_0) = \frac{\sum_{k=0}^{N-1} \left\| \theta_k - \hat{\theta}_k \right\|_{S_k}^2}{\left\| \theta_0 - \hat{\theta}_0 \right\|_{P_0^{-1}}^2 + \sum_{k=N-p}^{N-1} \left\| \varepsilon_{(k,\theta_k)} \right\|_{R_k^{-1}}^2} \tag{5}$$

with positive definite weighting matrices $P_0^{-1} \in \mathbb{R}^{n \times n}$ and $S_k \in \mathbb{R}^{n \times n}$. The notation $\|x\|_Q^2$ defines the weighted 2-norm of the vector x in the form $x^T Q x$. The direct minimization of the cost function (5) is not tractable, therefore, we choose a performance bound and seek an optimization strategy satisfying threshold [1,2]

$$V_1(\theta) < \frac{1}{\gamma} \tag{6}$$

where γ represents specified value tied up with restriction. Rearranging the cost function results in

$$V(\hat{\theta}_k, y_k, \theta_0) = -\frac{1}{\gamma} \left\| \theta_0 - \hat{\theta}_0 \right\|_{P_0^{-1}}^2 + \sum_{k=0}^{N-1} \left\| \theta_k - \hat{\theta}_k \right\|_{S_k}^2 - \frac{1}{\gamma} \sum_{k=N-p}^{N-1} \left\| y_k - \varphi_k^T \theta_k \right\|_{R_k^{-1}}^2 \tag{7}$$

This gaming issue is amenable to two solutions depending on the order of players. We choose a strategy in which the worst possible case is taken into account, ie., an upper value, for which the problem becomes

$$V(\hat{\theta}_k^*, y_k^*, \theta_0^*) = \min_{\hat{\theta}_k} \left\{ \max_{y_k, \theta_0} V(\hat{\theta}_k, y_k, \theta_0) \right\} \geq \max_{y_k, \theta_0} \left\{ \min_{\hat{\theta}_k} V(\hat{\theta}_k, y_k, \theta_0) \right\} \tag{8}$$

Because of mathematical optimization it is preferable to rewrite the last term in equation (7) as

$$\sum_{k=N-p}^{N-1} \|y_k - \varphi^T \theta_k\|_{R_k^{-1}}^2$$
$$= \sum_{k=0}^{N-1} \left[\left\| y_k - \varphi_k^T \theta_k \right\|_{R_k^{-1}}^2 - \left\| y_{k-p} - \varphi_{k-p}^T \theta_k \right\|_{R_{k-p}^{-1}}^2 \right] + \sum_{k=-p}^{-1} \left\| y_k - \varphi_k^T \theta_k \right\|_{R_k^{-1}}^2 \tag{9}$$

where zero values of variables $u(k)$, $y(k)$ are assumed, before starting the measurement in time $k = 0$. Substituting this formulation into equation (7) we obtain

$$V(\hat{\theta}_k, y_k, \theta_0)$$
$$= -\frac{1}{\gamma} \left\| \theta_0 - \hat{\theta}_0 \right\|_{P_0^{-1}}^2 + \sum_{k=0}^{N-1} \left[\left\| \theta_k - \hat{\theta}_k \right\|_{S_k}^2 - \frac{1}{\gamma} \left\| y_k - \varphi_k^T \theta_k \right\|_{R_k^{-1}}^2 + \frac{1}{\gamma} \left\| y_{k-p} - \varphi_{k-p}^T \theta_k \right\|_{R_{k-p}^{-1}}^2 \right]$$
$$= \psi_0 + \sum_{k=0}^{N-1} \mathcal{L}_k \tag{10}$$

where terms ψ_0 and $\mathcal{L}_k$ are defined by the equation above. To address minmax problem, the stationary point of V shall be found first with respect to θ_0 and then in accordance with y_k and $\hat{\theta}_k$. In order to maximize V with respect to θ_0 we transform the solution to dynamic optimization under constrain with restrictive condition based on dynamic equation of hypermodel (2)

$$V_\lambda(\hat{\theta}_k, y_k, \theta_0) = \psi_0 + \sum_{k=0}^{N-1}\left(\mathcal{L}_k + \frac{2}{\gamma}\lambda_{k+1}^T(\theta_k - \theta_{k+1})\right)$$
$$= \psi_0 + \sum_{k=0}^{N-1}\left(\mathcal{H}_k - \frac{2}{\gamma}\lambda_{k+1}^T\theta_k\right) + \frac{2}{\gamma}\lambda_0^T\theta_0 - \frac{2}{\gamma}\lambda_N^T\theta_N \tag{11}$$

where $2\lambda_{k+1}^T / \theta_k$ represents the time-varying scaled Lagrange multiplier [1,2]. The restrictive condition itself is not having an affect on the outcome of the solution. Hamiltonian is defined as

$$\mathcal{H}_k = \mathcal{L}_k + \frac{2}{\gamma}\lambda_{k+1}^T\theta_k \tag{12}$$

The constrained stationary point of V with respect to θ_0 shall probably be obtained from the solution of the three following equations

$$\frac{2\lambda_0}{\gamma} + \frac{\partial\psi_0}{\partial\theta_0} = 0 \tag{13}$$

$$\frac{2\lambda_N}{\gamma} = 0 \tag{14}$$

$$\frac{2\lambda_k}{\gamma} = \frac{\partial\mathcal{H}_k}{\partial\theta_k} \tag{15}$$

by including partial results into V and finding stationary points with respect to $\hat{\theta}_k$ and y_k, we obtain recursive relation for updating parameters $\hat{\theta}_k$ according to

$$P_{k+1} = P_k\left(I + \varphi_k R_k^{-1}\varphi_k^T P_k - \varphi_{k-p}R_{k-p}^{-1}\varphi_{k-p}^T P_k - \gamma S_k P_k\right)^{-1} \tag{16}$$

$$\hat{\theta}_{k+1} = \hat{\theta}_k + P_{k+1}\left[\varphi_k R_k^{-1}\left(y_k - \varphi_k^T\hat{\theta}_k\right) - \varphi_{k-p}R_{k-p}^{-1}\left(y_{k-p} - \varphi_{k-p}^T\hat{\theta}_k\right)\right] \tag{17}$$

If the previous relationship is to be seen as a solution of optimization problem, the following condition must hold at each step of calculation

$$\left(P_k^{-1} + \varphi_k R_k^{-1}\varphi_k^T - \varphi_{k-p}R_{k-p}^{-1}\varphi_{k-p}^T - \gamma S_k\right) > 0 \tag{18}$$

this condition is based on the request of positive definiteness of the second derivative with respect to $\hat{\theta}_k$, thus the found extreme will be a locally minimizing value of V. Equation (16) can be equivalently expressed by introducing intermediate covariance matrix P_{mk} as

$$P_{mk+1} = \left(P_k^{-1} + \varphi_k R_k^{-1}\varphi_k^T - \gamma S_k\right)^{-1} \tag{19}$$

$$P_{k+1} = \left(P_{mk+1}^{-1} - \varphi_{k-p}R_{k-p}^{-1}\varphi_{k-p}^T\right)^{-1} \tag{20}$$

substituting P_{mk} into equation (17) we obtain

$$\hat{\theta}_{k+1}$$
$$= \hat{\theta}_k + P_{k+1}P_{mk+1}^{-1}P_{mk+1}\varphi_k R_k^{-1}\left(y_k - \varphi_k^T\hat{\theta}_k\right) - P_{k+1}\varphi_{k-p}R_{k-p}^{-1}\left(y_{k-p} - \varphi_{k-p}^T\hat{\theta}_k\right)$$

$$= \hat{\theta}_k + \left[I - P_{k+1}\left(P_{k+1}^{-1} - P_{mk+1}^{-1}\right)\right]P_{mk+1}\varphi_k R_k^{-1}\left(y_k - \varphi_k^T\hat{\theta}_k\right) -$$
$$P_{k+1}\varphi_{k-p}R_{k-p}^{-1}\left(y_{k-p} - \varphi_{k-p}^T\hat{\theta}_k\right)$$

$$= \hat{\theta}_k + \left(I + P_{k+1}\varphi_{k-p}R_{k-p}^{-1}\varphi_{k-p}^T\right)P_{mk+1}\varphi_k R_k^{-1}\left(y_k - \varphi_k^T\hat{\theta}_k\right) - P_{k+1}\varphi_{k-p}R_{k-p}^{-1}\left(y_{k-p} - \varphi_{k-p}^T\hat{\theta}_k\right) \quad (21)$$

In order to find the recursive relation between $\hat{\theta}_k$ and $\hat{\theta}_{k+1}$ an intermediate estimate $\hat{\theta}_{mk}$ is defined

$$\hat{\theta}_{mk+1} = \hat{\theta}_k + P_{mk+1}\varphi_k R_k^{-1}\left(y_k - \varphi_k^T\hat{\theta}_k\right) \quad (22)$$

and resulting formulation for $\hat{\theta}_k$ is

$$\hat{\theta}_{k+1} = \hat{\theta}_{mk+1} - P_{k+1}R_{k-p}^{-1}\varphi_{k-p}\left[y_{k-p} - \varphi_{k-p}^T\hat{\theta}_k - \varphi_{k-p}^T P_{mk+1}\varphi_k R_k^{-1}\left(y_k - \varphi_k^T\hat{\theta}_k\right)\right]$$
$$= \hat{\theta}_{mk+1} - P_{k+1}\varphi_{k-p}R_{k-p}^{-1}\left(y_{k-p} - \varphi_{k-p}^T\hat{\theta}_{mk+1}\right) \quad (23)$$

In the case that $\gamma = 0$, equivalence of the SW-RecH$_\infty$ method according to presented equations (19), (20), (22), (23) and sliding window recursive weighted least-squares method (SW-RecWLS) is achieved.

3. SIMULATION RESULTS

The robustness of the proposed SW-RecH$_\infty$ method with regards to parameter estimation errors $\delta_k = \theta_k - \hat{\theta}_k$ will be investigated on the example simulation on the mathematical model.

Suppose the following time invariant dynamic process

$$A(q)y_k = B(q)u_k + C(q)v_k \quad (24)$$

$$A(q) = 1 + a_1q^{-1} + a_2q^{-2} + a_3q^{-3} = 1 - 2.6180q^{-1} + 2.2750q^{-2} - 0.6570q^{-3} \quad (25)$$

$$B(q) = b_1q^{-1} + b_2q^{-2} + b_3q^{-3} = 0.0002q^{-1} + 0.0009q^{-2} + 0.0002q^{-3} \quad (26)$$

$$C(q) = 1 + c_1q^{-1} + c_2q^{-2} = 1 + 0.60q^{-1} + 0.30q^{-2} \quad (27)$$

where q^{-1} is the shift operator in the meaning of $q^{-1}y_k = y_{k-1}$. Input to the process u_k is taken as a persistent excitation zero mean signal with variance $\sigma_u^2 = 1.00^2$, v_k, which represents exogenous stochastic disturbance term, is taken as a zero mean white noise sequence with variance $\sigma_v^2 = 0.02^2$. Although the discrete polynomial $C(q)$ causes, that disturbance term entering into a part $1\,/\,A(q)$ is colored, approximation of the process using ARX model will be realized. Applying the proposed method to estimate the ARX parameters of this process is evaluated as the 2-norm of the parameter estimation errors $\|\delta_k\|_2^2$ vs. step of calculation k in Fig. 1., where $\gamma = 0.001$, $p = 500$, the steady-state weighting matrices $R = 0.001^2$ and $S = I$.

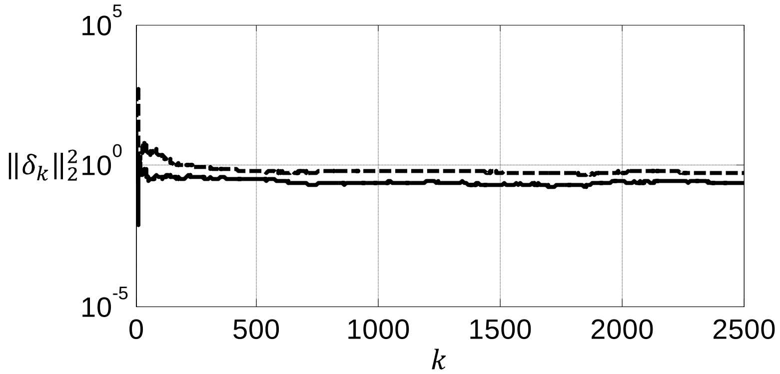

Figure 1: The parameter estimation errors δ_k vs. k. Solid line the SW-RecH$_\infty$ method, dashed line the SW-RecWLS. The SW-RecH$_\infty$ method performs about 82% better than the SW-RecWLS.

In relation to the previous example, the development of trace of covariance matrix $\mathrm{Tr}(P_k)$ depending on the length of the data horizon p is investigated. When the finite data technique is introduced, the steady-state trace of covariance matrix, and thus update gain, is bigger than in case that all data are incorporated into a calculation, as could be seen in Fig. 2.

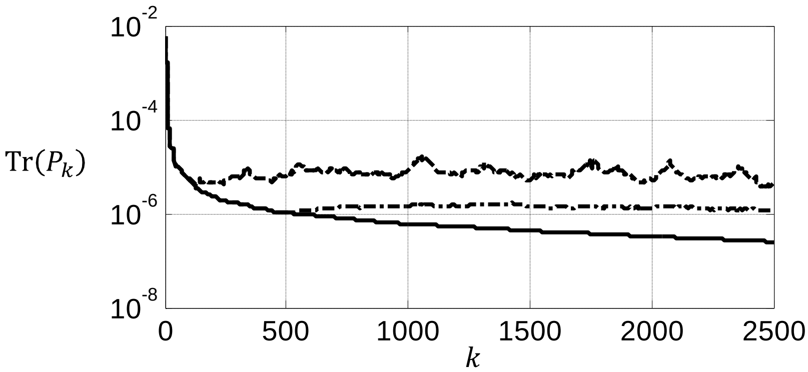

Figure 2: The trace of covariance matrix $\mathrm{Tr}(P_k)$ depending on the length of the data horizon p. Solid line $p = \infty$ (RecH$_\infty$), dash-dot line $p = 500$, dashed line $p = 100$.

4. CONCLUSION

In this paper, the sliding window recursive H$_\infty$ method (SW-RecH$_\infty$) over a finite data horizon through game theory approach is presented and derived. This method can be used to optimize a general linear hypermodel based on input-output data from the process. It was proved, that in the case of the unbounded cost function, which came from the game theory approach, the method is equivalent to the sliding window recursive weighted least-squares method (SW-RWLS). We can observe through the comparison of methods SW-RecH$_\infty$ and SW-RWLS, that the SW-RecH$_\infty$ is simply more robust variant of the SW-RWLS method and provides more accurate estimates in the case, where there is insufficient information about stochastic properties of the process.

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